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Brane-world cosmology

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Abstract: A simple model of the brane-world cosmology has been proposed, which is characterized by four parameters, namely, the bulk cosmological constant, the spatial curvature of the universe, the radiation strength arising from bulk space-time and the breaking parameter of $\mathbb{Z}_2$-symmetry. The bulk space-time is assumed to be locally static five-dimensional analogue of the Schwarzschild-Anti-de Sitter space-time, and then the location of three-brane is determined by the metric junction. The resulting Friedmann equation recovers the standard cosmology, and a new term arises if $\mathbb{Z}_2$-symmetry is not assumed, which behaves as a cosmological term in the early universe, next turns to a negative curvature term, and finally damps rapidly.

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Recently, Randall and Sundrum have proposed two models in which our universe is a three-brane imbedded in a five-dimensional Anti-de Sitter space AdS_5. Their first model is a possible solution to the hierarchy problem between the weak and Planck scales, in which the fifth dimension is bounded by two domain walls with positive and negative tension, and the large mass hierarchy can be explained in terms of the exponential factor of the AdS_5 metric along the extra dimension. In this model, the visible world is assumed to be the negative-tension brane. However, Shiromizu et al. have shown that the gravitation is repulsive on such a brane, so that this model is physically inacceptable. In the second model, they suggested that the extra dimension need not be compact, and that it is sufficient to put a single positive-tension brane. They have also shown that the zero-mode of the linear metric perturbation recovers Newton's inverse-square law and that the Kaluza-Klein modes make only small contribution to the gravitaion on the brane. Currently, there seems to be no strong evidence to exclude this model. On the other hand, it is also of theoretical interest to seek for the cosmological and astrophysical prediction of the brane-world senario. Especially, the black holes [5, 6] and expanding universe [7]–[17] have been researched by many authors. As far as a black hole solution is concerned, it is only known for a (2+1)+1-dimensional bulk space-time [6], so that a (3+1)+1-dimensional solution is desired. While for the expanding universe, many authors have derived the evolution equation for the metric variables, there are a few versions of explicit bulk solutions. Moreover, the known bulk solutions have rather a complicated expression, and seem to be difficult to be handled. To test the brane-world scenario observationally, it would be necessary to know the evolution of the density fluctuation. The density fluctuation cannot be determined only by the background metric of the three-brane. We should also know the five-dimensional metric, since the electric part of the five-dimensional Weyl tensor contributes to the gravitational field on the three-brane through the four-dimensional Einstein equation [3]. However, as mensioned above, though Friedmann equations have been derived for several cases, only a few global solutions are known. So we shall attempt to construct a simple model of brane-world cosmology. The usual approach has been as follows: assume the appropriate form of the metric which is *manifestly $\mathbb{Z}_2$-symmetric*, and solve the junction condition and the Einstein equation. Alternatively, here we assume an appropriate five-dimensional metric at the beginning, and then find the location of the three-brane.

Let us consider the five-dimensional bulk space-time. In their original scenario, Randall and Sundrum take AdS_5 as the bulk space-time. Here we assume the bulk space-time is to be Einstein space with negative cosmological constant. As such space, we take the five-dimensional analogue of the Schwarzschild-Anti-de Sitter space-time in the following form:

$$\overset{5}{g} = -h(a)dt^2 + \frac{1}{h(a)}da^2 + a^2\left[d\chi^2 + f_k(\chi)^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2)\right], \quad (k = 0, \pm 1), \quad (1)$$

where $f_0(\chi) = \chi$, $f_1(\chi) = \sin\chi$ and $f_{-1}(\chi) = \sinh\chi$. The solution of field equation $\overset{5}{R}_{\mu\nu}=\overset{5}{\Lambda}\overset{5}{g}_{\mu\nu}$ with the cosmological constant $\overset{5}{\Lambda}= -4l^2$ is given by

$$h(a) = k - \frac{\alpha}{a^2} + l^2 a^2 \,, \tag{2}$$

where α and l are constants. If $\alpha = 0$, then this reduces to AdS_5, however, $\alpha \neq 0$ generates the electric part of the Weyl tensor. The three-brane model can be obtained in the similar manner to the spherical thin-shell model in four dimension [18], i.e. by patching together two manifolds, which may have a different parameter α from each other, but have common k and l, at the timelike hypersurface.

We shall consider the location of three-brane in the form $t = t(\tau)$, $a = a(\tau)$ parametrized by the proper time τ on the brane. Then the induced metric of three-brane will be given by

$$\overset{4}{g}= -d\tau^2 + a(\tau)^2 \left[d\chi^2 + f(\chi)^2(d\vartheta^2 + \sin^2\vartheta\varphi^2)\right] , \tag{3}$$

i.e. τ and $a(\tau)$ correspond to the cosmic time and scale factor of Friedmann-Robertson-Walker universe, respectively. The tangent vector of the brane can be written in the form

$$u = \dot{t}_{R,L}\frac{\partial}{\partial t_{R,L}} + \dot{a}\frac{\partial}{\partial a} \,, \tag{4}$$

where the dot denotes the derivative with respect to the proper time τ, and the subscripts R and L denote the quantity evaluated on the right and the left side of the brane, respectively (note that the scale factor $a(\tau)$ and the proper time τ are common on both sides). Given the tangent vector u, the normal 1-form to the three-brane n is determined up to the sign. Here we fix this freedom by setting

$$n = \dot{a}dt_R - \dot{t}_R da = -\dot{a}dt_L + \dot{t}_L da \,. \tag{5}$$

This choice gives the Randall-Sundrum model in the limit $\alpha_{R,L} \to 0$ (other choices are possible, though the resulting Friedmann equation is not affected).

The embedding of three-brane is characterized by the induced metric $\overset{4}{g}$ and the extrinsic curvature $K_{R,L}$ on both sides

$$\overset{4}{g}_{\mu\nu}=\overset{5}{g}_{\mu\nu} - n_\mu n_\nu \,, \tag{6}$$

$$K_{\mu\nu} =\overset{4}{g}_{\mu}{}^{\lambda}\nabla_\lambda n_\nu \,, \tag{7}$$

where ∇ denotes the Riemannian connection compatible with $\overset{5}{g}$. The extrinsic curvature has the following non-vanishing components:

$$(K_R)_{\mu\nu}u^\mu u^\nu = (h_R\dot{t}_R)^{-1}(\ddot{a} + h'_R/2) \,, \tag{8}$$

$$(K_R)^\chi_\chi = (K_R)^\vartheta_\vartheta = (K_R)^\varphi_\varphi = -h_R\dot{t}_R/a \,, \tag{9}$$

$$(K_L)_{\mu\nu}u^\mu u^\nu = -(h_L\dot{t}_L)^{-1}(\ddot{a} + h'_L/2) \,, \tag{10}$$

$$(K_L)^\chi_\chi = (K_L)^\vartheta_\vartheta = (K_L)^\varphi_\varphi = h_L\dot{t}_L/a \,, \tag{11}$$

where $h_{R,L} = k - \alpha_{R,L}/a^2 + l^2a^2$. Thus the extrinsic curvature has the jump at the brane. This jump can be interpreted as generated by the matter field confined to the brane. The surface stress-energy tensor on the brane S is defined by

$$S_{\mu\nu} = -\frac{1}{\kappa^2}\left\{K_{\mu\nu} - K^\lambda_\lambda \overset{4}{g}_{\mu\nu}\right\}^- , \tag{12}$$

where for any tensor field Q, $\{Q\}^- = Q_R - Q_L$ is defined. In the present case, S necessary has the ideal-fluid form

$$S_{\mu\nu} = (\varrho + p)u_\mu u_\nu + p\,\overset{4}{g}_{\mu\nu}\,, \tag{13}$$

due to the spatial symmetry of the three-section generated by Span$\{d\chi, d\vartheta, d\varphi\}$, where ϱ and p are the energy density and pressure of the brane, respectively, or equivalently,

$$\{K_{\mu\nu}\}^- = -\kappa^2\left(S_{\mu\nu} - \frac{1}{3}S^\lambda_\lambda \overset{4}{g}_{\mu\nu}\right) = -\kappa^2\left[(\varrho+p)u_\mu u_\nu + \frac{\varrho}{3}\overset{4}{g}_{\mu\nu}\right]. \tag{14}$$

The Gauss-Codazzi equations lead to the following metric junction conditions [18]:

$$\frac{\kappa^2}{2}[(K_R)_{\mu\nu} + (K_L)_{\mu\nu}]S^{\mu\nu} = \left\{\overset{5}{G}_{\mu\nu}\, n^\mu n^\nu\right\}^- , \tag{15}$$

$$\kappa^2\, \overset{4}{g}_{\mu}{}^{\lambda}\nabla_\nu S_\lambda{}^\nu = \overset{4}{g}_{\mu}{}^{\lambda}\left\{\overset{5}{G}_\lambda{}^\nu n_\nu\right\}^- . \tag{16}$$

The last equation is the conservation law, which in the present case leads to a familiar equation

$$\frac{d}{d\tau}(\varrho a^3) + p\frac{d}{d\tau}(a^3) = 0\,. \tag{17}$$

On the other hand, the explicit forms of eqs. (14) and (15) are

$$(h_R\dot{t}_R)^{-1}\left(\ddot{a} + \frac{h'_R}{2}\right) + (h_L\dot{t}_L)^{-1}\left(\ddot{a} + \frac{h'_L}{2}\right) = -\kappa^2\left(p + \frac{2}{3}\varrho\right), \tag{18}$$

$$h_R\dot{t}_R + h_L\dot{t}_L = \frac{\kappa^2}{3}\varrho a\,, \tag{19}$$

$$(h_R\dot{t}_R)^{-1}\left(\ddot{a} + \frac{h'_R}{2}\right) - (h_L\dot{t}_L)^{-1}\left(\ddot{a} + \frac{h'_L}{2}\right) = 3\frac{p}{\varrho a}(h_R\dot{t}_R - h_L\dot{t}_L)\,, \tag{20}$$

which lead to

$$h_R\dot{t}_R = -\frac{3}{4\kappa^2\varrho}(h'_R - h'_L) + \frac{\kappa^2}{6}\varrho a\,, \tag{21}$$

$$h_L\dot{t}_L = \frac{3}{4\kappa^2\varrho}(h'_R - h'_L) + \frac{\kappa^2}{6}\varrho a\,, \tag{22}$$

and the equation of motion

$$\ddot{a} = \frac{27(\alpha_R - \alpha_L)^2}{4\kappa^4} \frac{p}{\varrho^3 a^7} - \frac{\alpha_R + \alpha_L}{2a^3} - \frac{\kappa^4}{18} \varrho \left(\varrho + \frac{3}{2} p \right) a - l^2 a \, . \tag{23}$$

The first integral of this equation, namely the Friedmann equation, can be obtained by putting eq. (21) into the normalization condition $-h_R \dot{t}_R^2 + \dot{a}^2/h_R = -1$, and we have

$$\left(\frac{\dot{a}}{a} \right)^2 = -\frac{k}{a^2} - l^2 + \frac{\kappa^4}{36} \varrho^2 + \frac{\alpha_R + \alpha_L}{2a^4} + \frac{9(\alpha_R - \alpha_L)^2}{4\kappa^4 \varrho^2 a^8} \, . \tag{24}$$

This result is in agreement with Binétruy et al. [15] except the last term presented here. This new term arises due to the asymmetry between the left and right world, which vanishes in the $\mathbb{Z}_2$-symmetric case $\alpha_R = \alpha_L$. According to the brane-world scenario, it is natural to assume that the matter component consists of vacuum energy and the ordinary matter, namely

$$\varrho = \rho + \sigma \, , \tag{25}$$

$$p = P - \sigma \, , \tag{26}$$

where ρ and P denote the energy density and pressure of ordinary matter, respectively, and $\sigma =$ constant is the tension of the brane. Then the conservation eq. (16) and the Friedmann eq. (24) become

$$\frac{d}{d\tau}(\rho a^3) + P \frac{d}{d\tau}(a^3) = 0 \, , \tag{27}$$

$$\left(\frac{\dot{a}}{a} \right)^2 = \frac{8\pi G_N}{3} \rho - \frac{k}{a^2} + \frac{\Lambda}{3} + \frac{\alpha_R + \alpha_L}{2a^4} + \frac{\kappa^4}{36} \rho^2 + \frac{9(\alpha_R - \alpha_L)^2}{4\kappa^4 (\rho + \sigma)^2 a^8} \, , \tag{28}$$

where

$$8\pi G_N = \frac{\kappa^4 \sigma}{6} \, , \tag{29}$$

$$\Lambda = \frac{\kappa^4 \sigma^2}{12} - 3l^2 \tag{30}$$

are regarded as four-dimensional Newton's constant of gravitation and the cosmological constant, respectively. Equation (28) recovers the standard cosmology in a good approximation for appropriate ranges of parameters. The first three terms on r.h.s. are usual appear in standard cosmology. The forth term behaves like radiation. However, this can be negative in the present model. The origin of this term is now clarified, namely the electric (Coulomb) part of the Weyl tensor in the five-dimensional back ground metric. The fifth term rapidly damps in most cases of interest, e.g. $\propto a^{-6}$ for dust, and $\propto a^{-8}$ for radiation, so that this term becomes significant in the early universe. Binétury et al. [15] have discussed a constraint on this term by nucleosynthesis. The last term, which is present if the assumption of

$\mathbb{Z}_2$-symmetry is dropped, behaves in various ways. When $\rho \ll \sigma$, it behaves like positive Λ-term for radiation, so that we might have a inflation without any other fields in the early universe. For dust, it behaves like negative curvature term, and eventually, when $\sigma > \rho$, it will rapidly damp.

Thus, our model seems to give us many features of the brane-world cosmology. In fact, there are 6 cosmological parameters corresponding to 6 terms in r.h.s. of eq. (28), so that one might apply this model to the dark matter, or cosmological constant problem. The bulk space-time is given in the simplest form imaginable, and it is easy to handle this model. Therefore the calculation of the density fluctuation or gravitational radiation would be possible, which are of interest from the viewpoints of observational cosmology and astrophysics.

Note added in proof. After this work is completed, I have received refs. [19, 20] which overlap with the subject of this work, and where similar results were found.

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